Lecture 5:

Coded Aperture Design Using Sphere Packing

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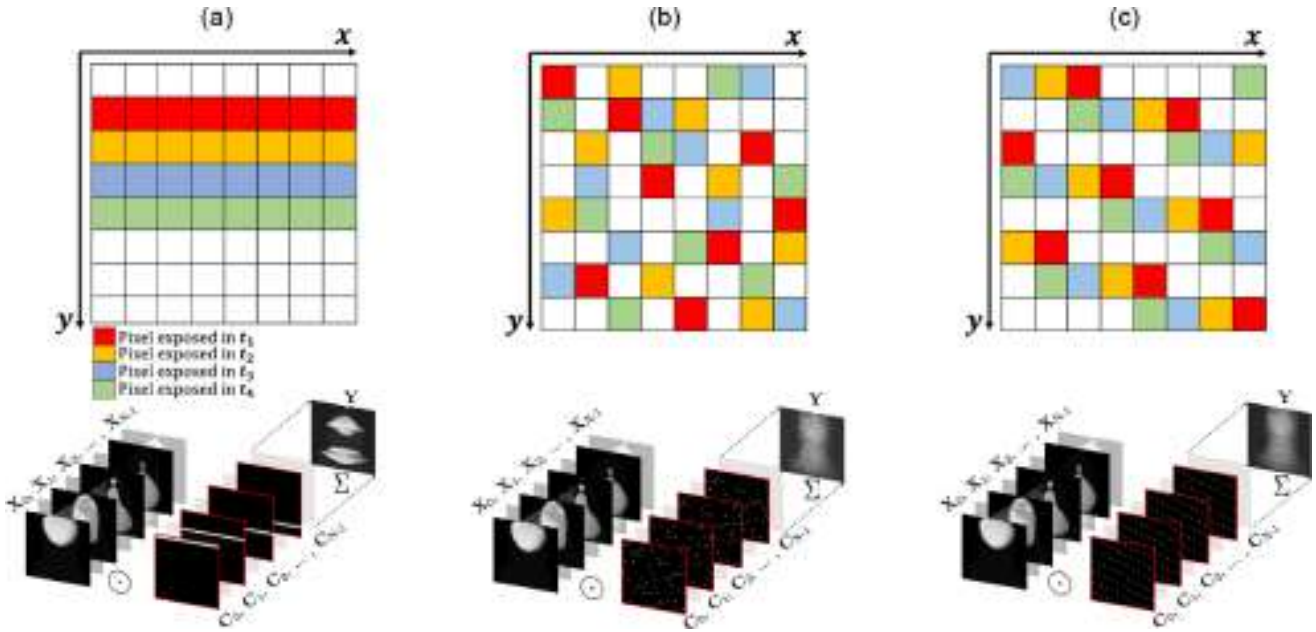
Valparaíso, Chile



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Motivation: Snapshot Video

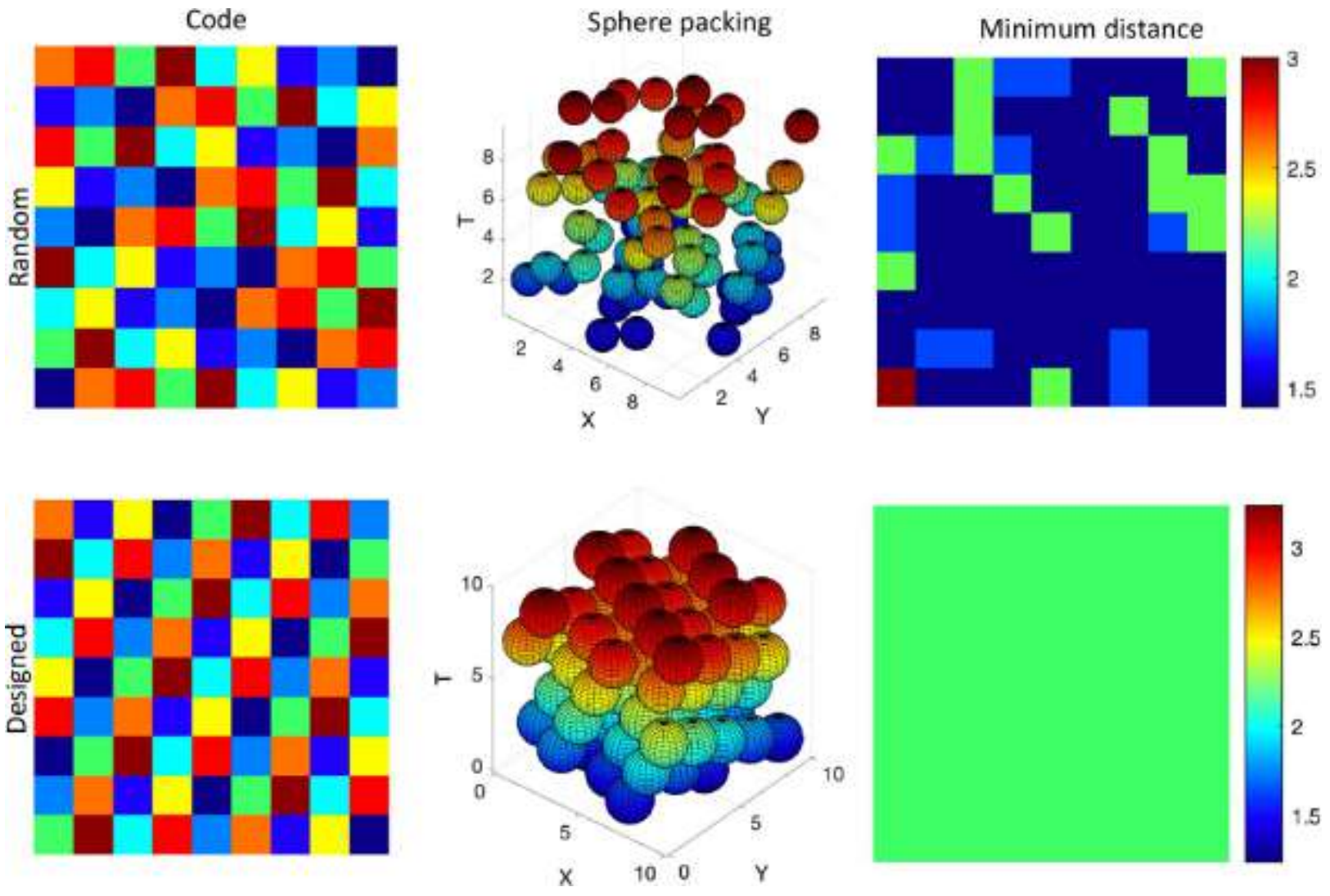
* Shuffled rolling shutter $f(x, y, t) \in \mathbb{R}^3$ snapshot video[1].



Shuffled rolling shutter [2].

[3] E. Vera; F. Guzman; N. Diaz, "Shuffled Rolling Shutter for Snapshot Temporal Imaging", Opt. Express, Vol. 30, pp.887-901,2022.

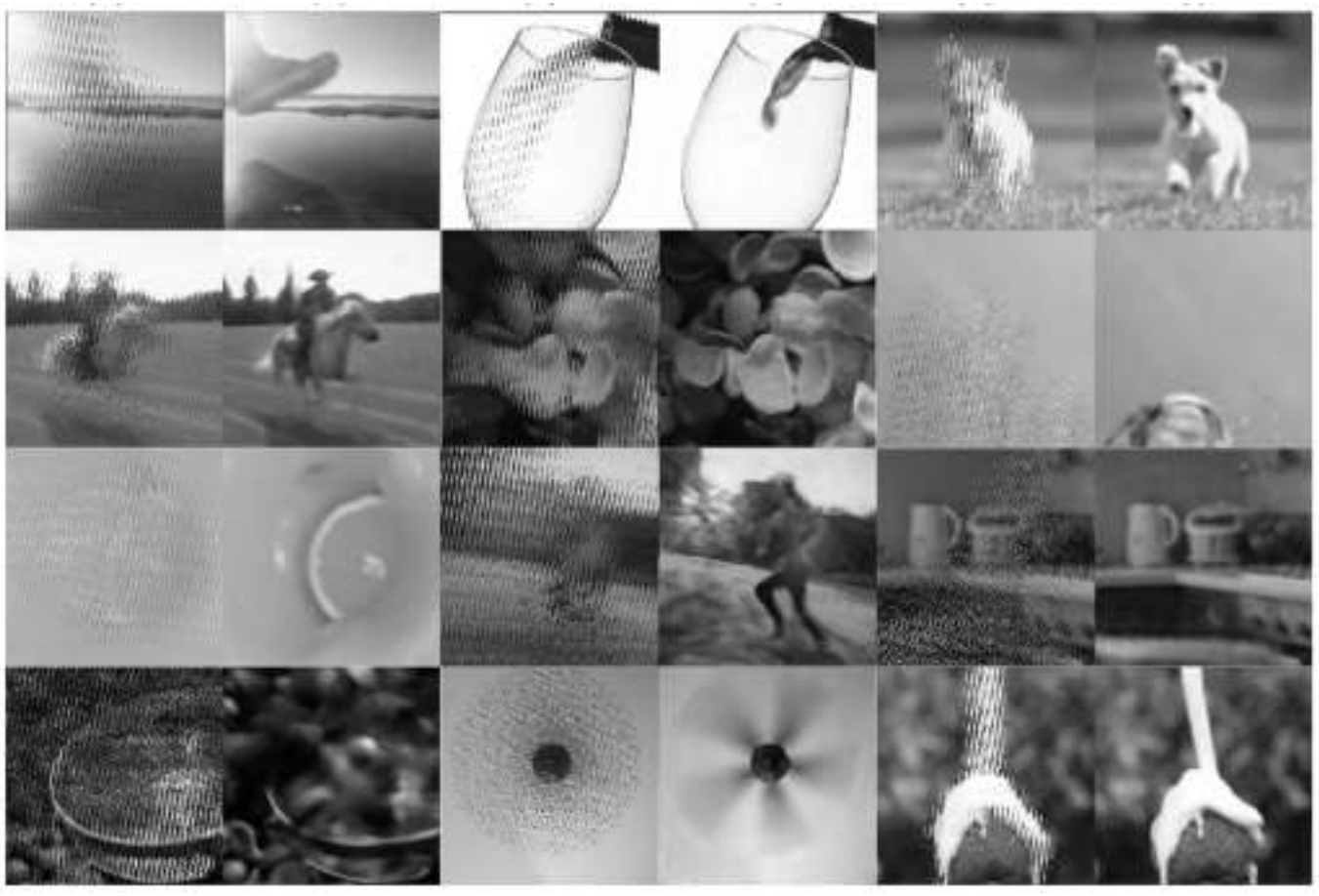
Motivation: Snapshot Video



Sphere Packing Optimization, random against designed.

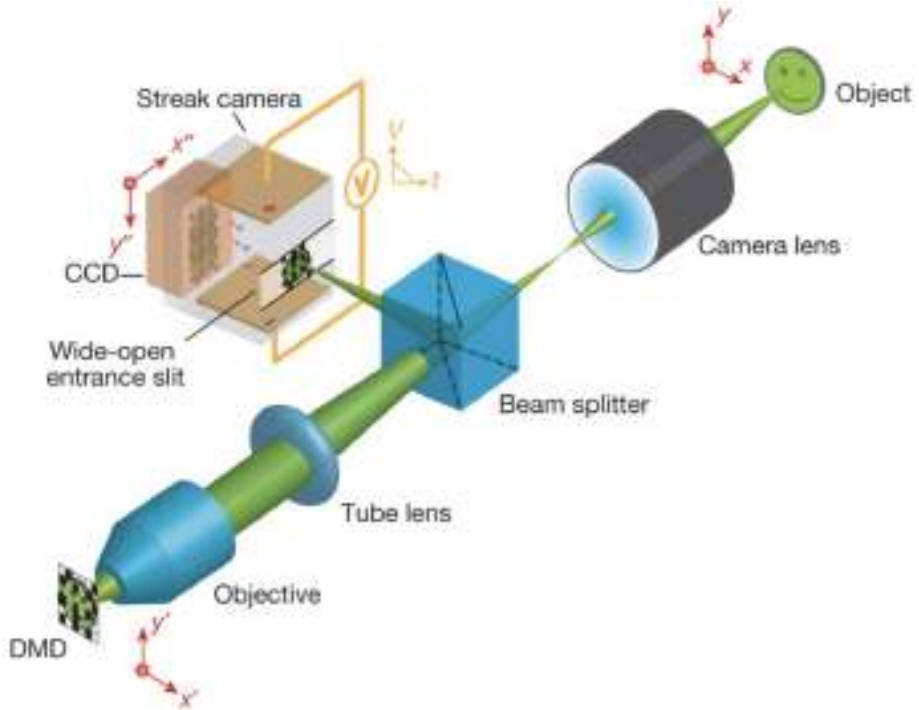
Motivation: Snapshot Video

* Video Reconstructions of Shuffled rolling shutter $f(x, y, t) \in \mathbb{R}^3$



Compressive Ultrafast Photography (CUP)

Compressive Ultrafast Photography (CUP) captures compressive projections of video. The scene is encoded using a binary coded aperture. Then, the encoded scene is sheared by the temporal shearing operator. The encoded and sheared scene is captured in the detector.



Compressive Ultrafast Photography (CUP) scheme [4].

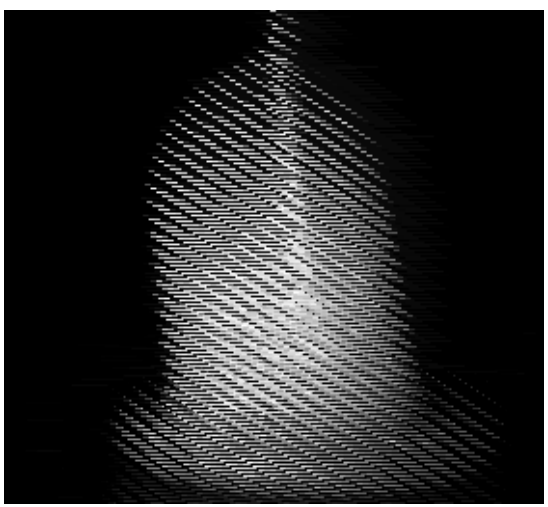
[4] Gao, L., Liang, J., Li, C. et al. Single-shot Compressed Ultrafast Photography at one Hundred Billion Frames per Second. Nature 516, 74–77 (2014).

Compressive Ultrafast Photography: Discrete Model

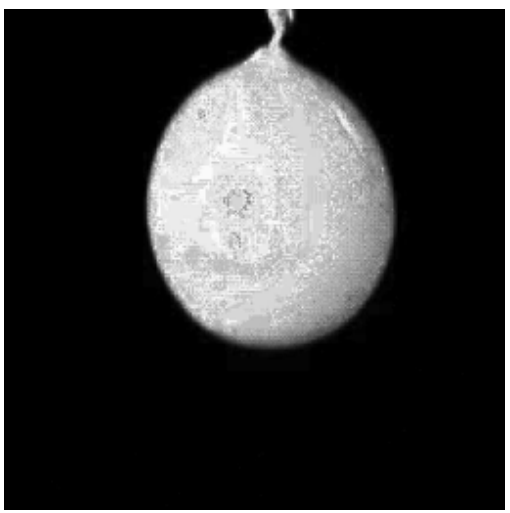
The corresponding discrete model is as follows:

$$Y_{(:,\bar{t}\,)} = \sum_{t=0}^{T-1} \mathcal{X}_{(:,:,\bar{t}\,)} \odot \mathcal{C}_{(:,:,\bar{t}\,)} + \mathbf{\Omega}, \tag{1}$$

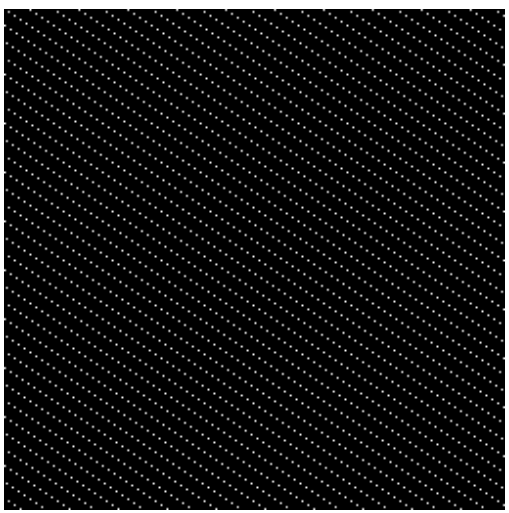
where $\mathcal{X} \in \mathbb{R}^{M \times N \times T}$ is the tensor that represents the 3-dimensional video datacube, and $\mathcal{C} \in \mathbb{R}^{M \times N \times T}$ denotes the tensor of the $3D$-CA, where $\bar{t}$ denotes the temporal shearing of CUP setup.



measurement $\mathbf{Y}_{(:,\bar{t})}$



video $\mathcal{X}_{(:,:,t)}$



coded aperture $\mathcal{C}_{(:,:,t)}$

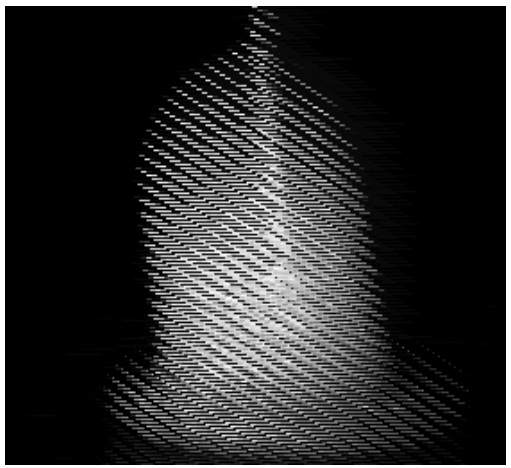
Reconstruction Algorithm

We start by expanding the measurement $\mathbf{Y}$ into the datacube $\bar{\mathcal{X}}_{(:,:,t)}$ by using the CA $\mathcal{C}_{(:,:,\bar{t})}$ such that

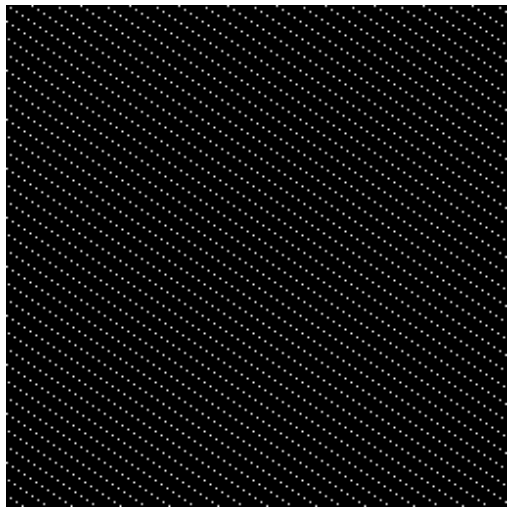
$$\bar{\mathcal{X}}_{(:,:,t)} = \mathcal{C}_{(:,:,\bar{t})} \odot \mathbf{Y}_{(:,\bar{t}\,)}. \tag{2}$$



video $\hat{\mathcal{X}}_{(:,:,t)}$.



measurement $\mathbf{Y}_{(:,:)}$.



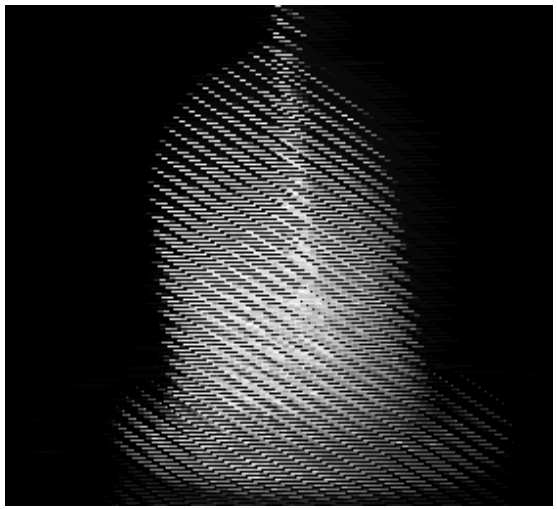
coded aperture $\mathcal{C}_{(:,:,\bar{t}\,)}$.

The algorithm to recover the underlying datacube is Nearest Neighbor Interpolation (NNI)[5], whose input is $\bar{\mathcal{X}}_{(:,:,t)}$ and its output is $\hat{\mathcal{X}}_{(:,:,t)}$.

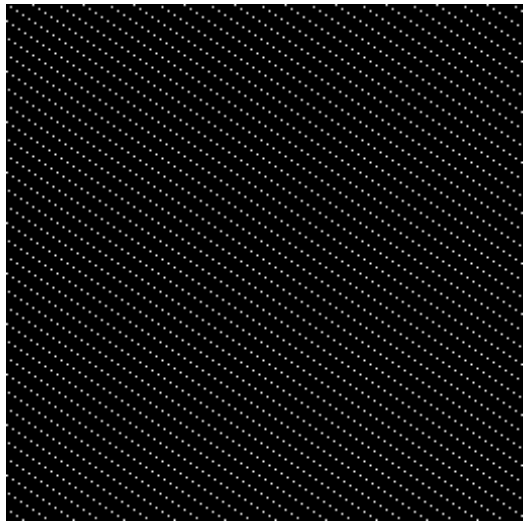
[5] Amidror, Isaac, "Scattered Data Interpolation Methods for Electronic Imaging Systems: a Survey.", J. Electronic Imaging, Vol. 11, pp.157-176, 2002.

Comparison of Image Quality Reconstruction: 25 frames

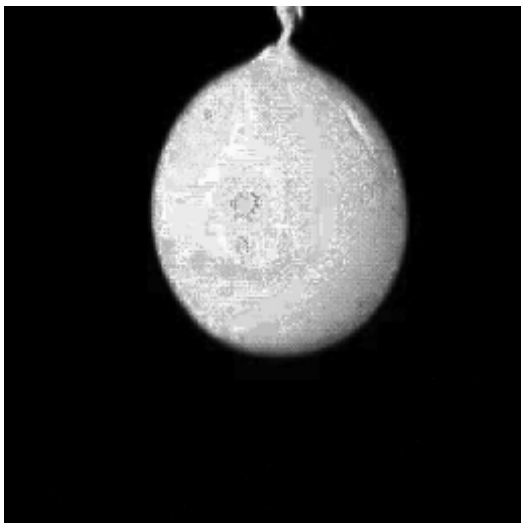
Measurement $\mathbf{Y} \in \mathbb{R}^{256 \times 280}$



Coded aperture $\hat{\mathcal{C}}$, $Tr = 1/25$.



Groundtruth.



$\mathcal{X} \in \mathbb{R}^{256 \times 256 \times 25}$

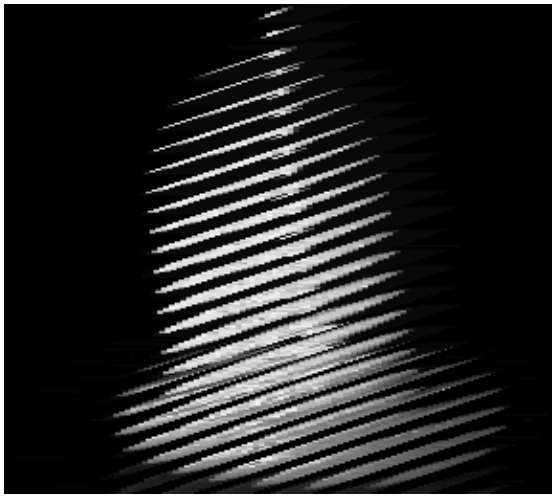
Video Reconstruction $\hat{\mathcal{X}} \in \mathbb{R}^{256 \times 256 \times 25}$.



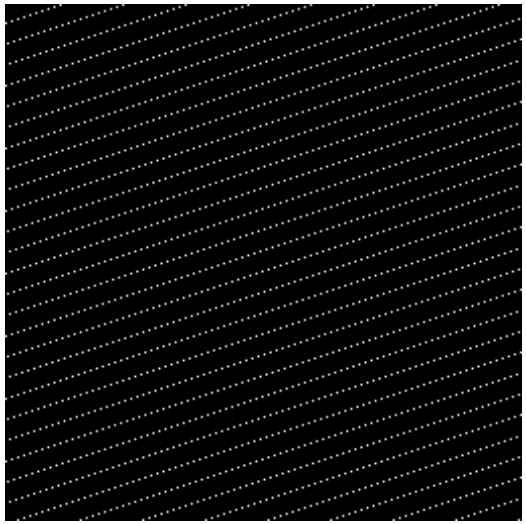
PSNR 28.04 [**dB**], SSIM 0.68.

Comparison of Image Quality Reconstruction: 31 frames

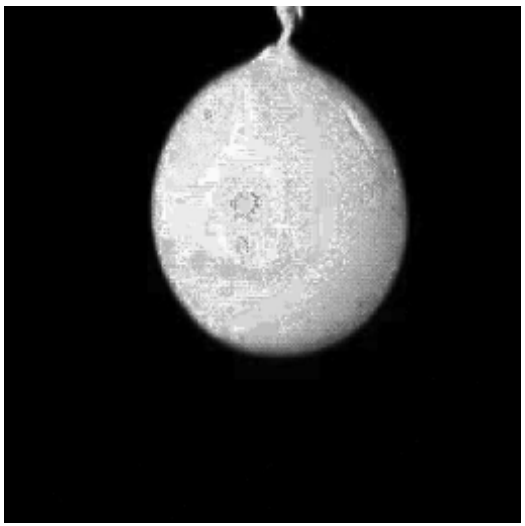
Measurement $\mathbf{Y} \in \mathbb{R}^{256 \times 286}$



Coded aperture $\hat{\mathcal{C}}$, $Tr = 1/31$.



Groundtruth.



$\mathcal{X} \in \mathbb{R}^{256 \times 256 \times 31}$

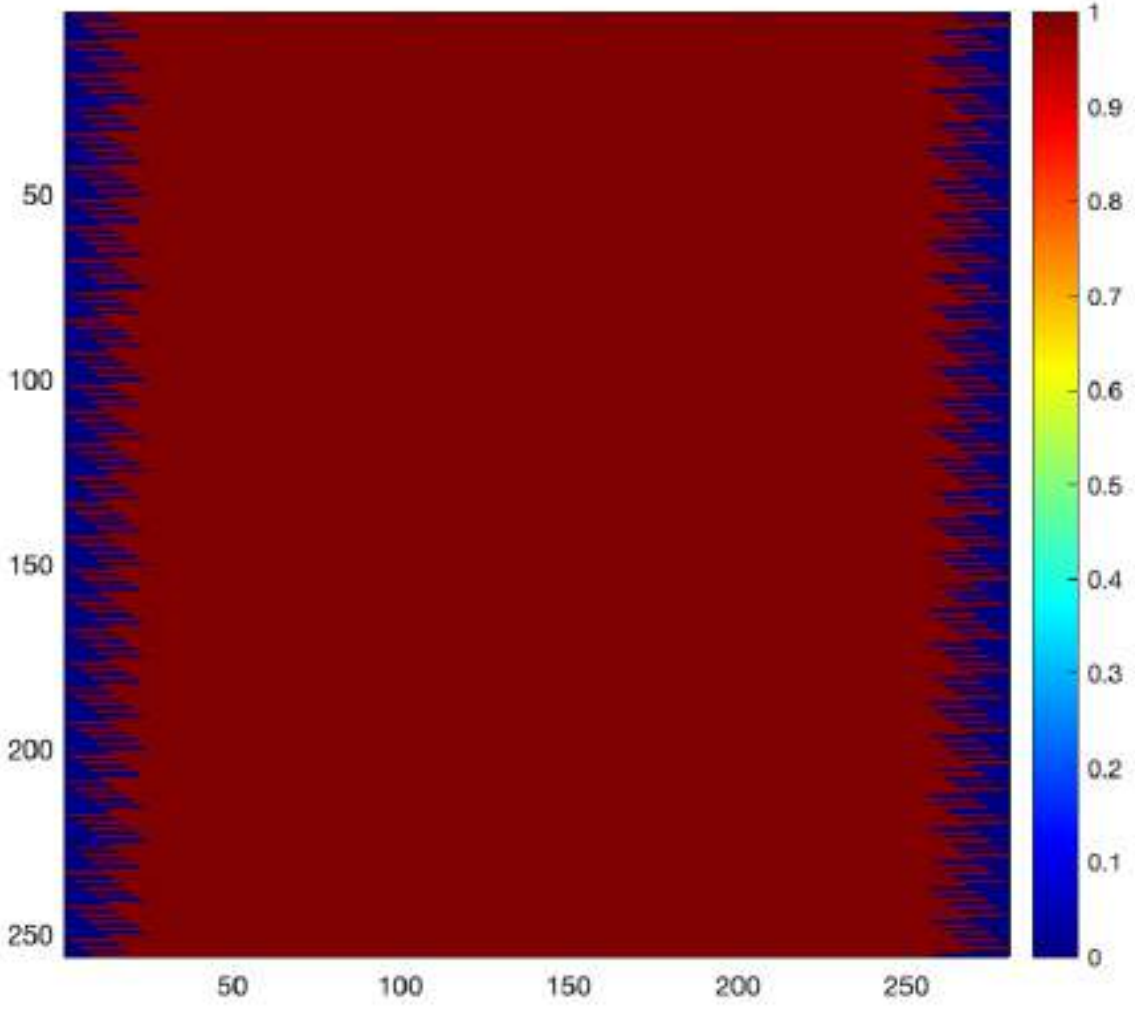
Video Reconstruction $\hat{\mathcal{X}} \in \mathbb{R}^{256 \times 256 \times 31}$.



PSNR 26.79 [**dB**], SSIM 0.64.

Why Sphere Packing Works in CUP?

Sphere packing promotes uniform sampling in CUP and coded aperture snapshot spectral imaging (CASSI) architectures.

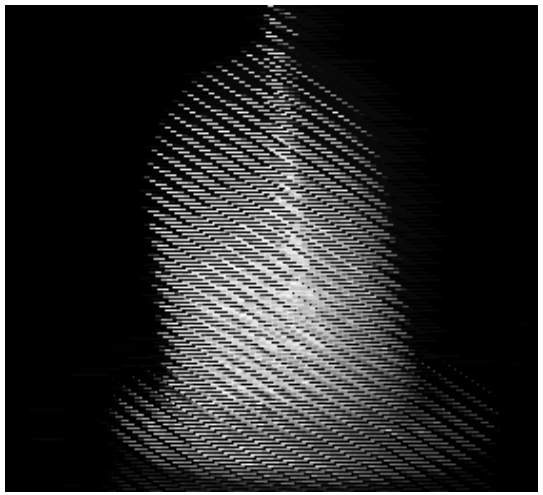


Conclusions

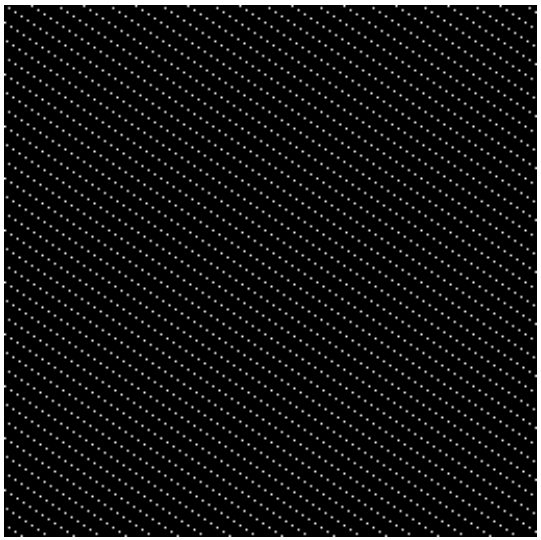
* We introduced a novel compressive ultrafast photography approach that exploits optimal sphere packing.
* Our approach is able to accurately recover a video from a compressive single snapshot.
* The proposed approach obtains image reconstruction quality up to 28.04 [dB] of PSNR and 0.68 of SSIM.



Video reconstruction $\hat{\mathcal{X}}_{(:,:,t)}$.



measurement $\mathbf{Y}_{(:,:)}$.



coded aperture $\mathcal{C}_{(:,:,t)}$.

What is Sphere Packing?

The sphere packing problem asks for the densest packing of $\mathbb{R}^n$ with congruent balls. Equivalent to answer the question:

What is the largest fraction of $\mathbb{R}^n$ that can be covered by congruent balls with disjoint interiors?



Sphere Packing Density

One-dimensional sphere packing is boring:

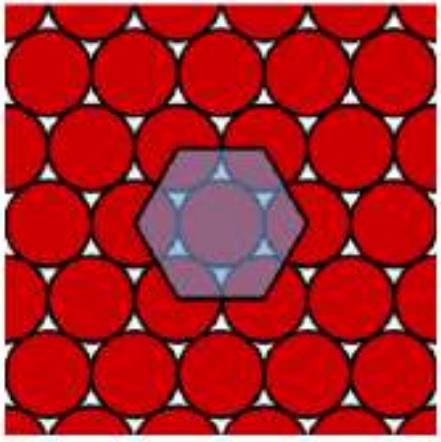
1D



density=1

Two-dimensional sphere packing is more interesting and attractive

2D



density=$\frac{\pi}{\sqrt{12}} \approx 0.9068$

Three dimensions strains human ability to prove

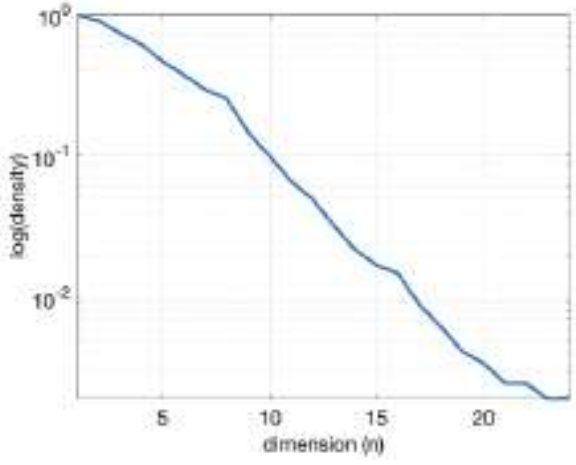
3D



density=$\frac{\pi}{\sqrt{18}} \approx 0.7404$

What happens in four dimensions?

Sphere Packing Density in $\mathbb{R}^n$ and Applications



n	density	n	density	n	density	n	density
1	1.00000	7	0.29529	13	0.03201	19	0.00412
2	0.90689	8	0.25366	14	0.02162	20	0.00339
3	0.74048	9	0.14577	15	0.01685	21	0.00246
4	0.61685	10	0.09961	16	0.01470	22	0.00245
5	0.46525	11	0.06623	17	0.00881	23	0.00190
6	0.37294	12	0.04945	18	0.00616	24	0.00192

Sphere packing density in $\mathbb{R}^n$ [6]. Optimal density in blue color.

Applications in Computational Imaging

[3] H. Cohn, "A Conceptual Breakthrough in Sphere Packing", Notices of the American Mathematical Society, Vol. 64, pp.102-115, 2017.

Applications

* Compressive video[4] , [5] $f(x, y, t) \in \mathbb{R}^3$ and compressive spectral imaging[6] $f(x, y, \lambda) \in \mathbb{R}^3$ are sampling problems in 3D.
* Compressive spectral depth $f(x, y, z, \lambda) \in \mathbb{R}^4$ and compressive spectral video $f(x, y, \lambda, t) \in \mathbb{R}^4$ are sampling problems in 4D.
* Compressive spectral light field samples a function $f(x, y, z, \theta, \phi) \in \mathbb{R}^5$, is a problem in 5D.
* Sampling the plenoctic function involves sensing in 7D $f(x, y, z, \theta, \phi, \lambda, t) \in \mathbb{R}^7$ being (x, y, z) 3D-space dimensions, (λ) spectral dimension, (θ, ϕ) two angular dimensions, and (t) time.

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[4] E. Vera; F. Guzman; N. Diaz, "Shuffled rolling shutter for snapshot temporal imaging", Opt. Express, Vol. 30, pp.887-901,2022.

[5] A. Alvarado; N. Díaz; P. Meza; F. Guzman, E. Vera; , "Multispectral mosaic design using a sphere packing filter array", in imaging and Applied Optics Congress 2022.

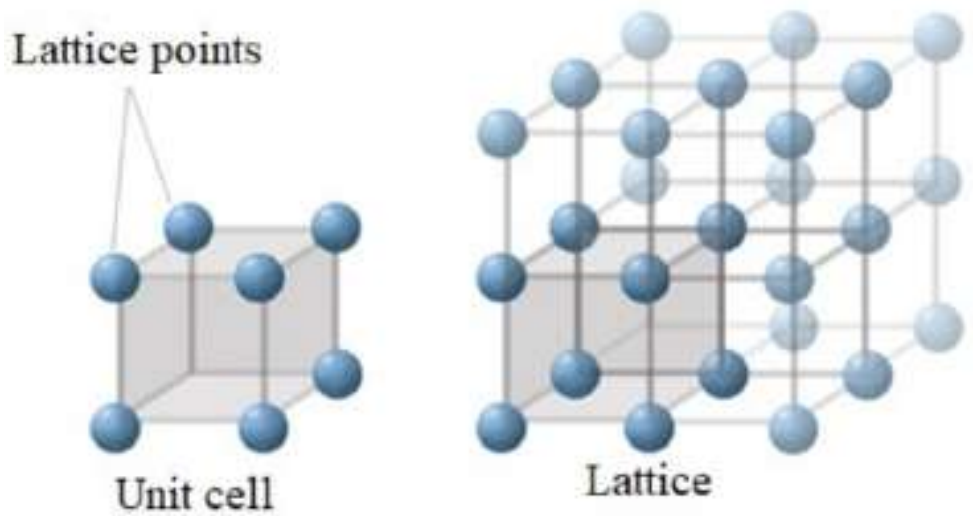
[6] N. Diaz, A. Alvarado, P. Meza, F. Guzmán and E. Vera, "Multispectral Filter Array Design by Optimal Sphere Packing," in IEEE Transactions on Image Processing, vol. 32, pp. 3634-3649, 2023, doi: 10.1109/TIP.2023.3288414.

What is a Lattice?

A typical lattice $\mathbf{\Lambda} \in \mathbb{R}^n$ thus has the form

$$\mathbf{\Lambda} = \sum_{i=1}^{n} a_i v_i | a_i \in \mathbb{Z} \tag{3}$$

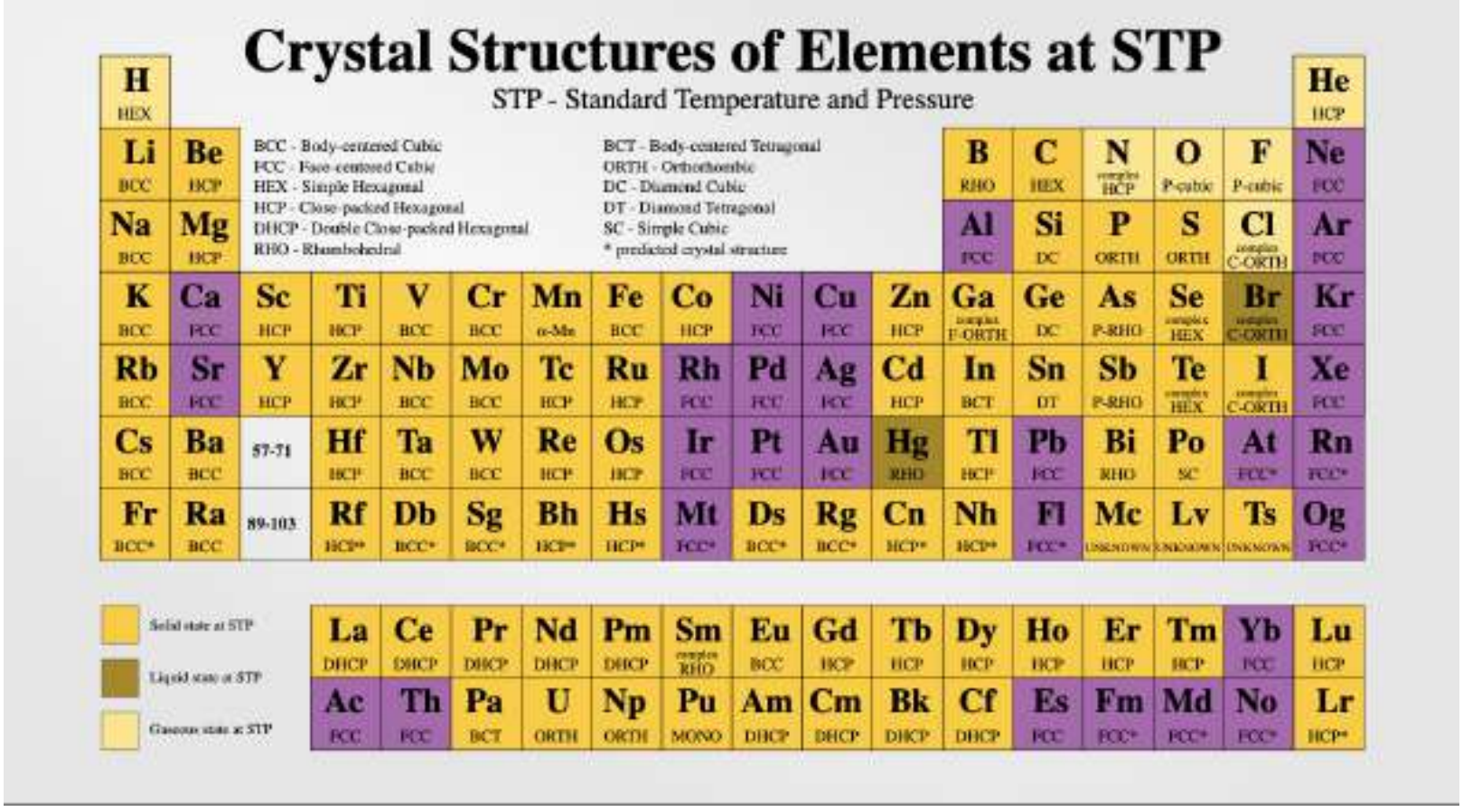
* where $\mathbf{M} = [v_1, \ldots, v_n]$ is a unit cell or Generator Matrix basis in $\mathbb{R}^n$
* The Gram Matrix $\mathbf{A} = \mathbf{M}^T\mathbf{M}$. Its entries (i, j) are given by $\langle v_i, v_j \rangle$ [7].



$\mathbf{M}$ is a unit cell and $\mathbf{\Lambda}$ is a lattice.

[7] Conway, J. and Sloane, N.J.A, "Sphere Packings, Lattices and Groups", Springer New York, 2013.

Face Centered-Cubic Lattice in Chemistry and Physics



[7] https://msestudent.com/face-centered-cubic-fcc-unit-cell/.

History of the Sphere Packing Problem



- 1611
- Johannes **Kepler conjectured** about the closest packing of equal spheres.
- He did not have a prove to the conjecture.



- 1998
- Thomas Hales provides the formal proof of Kepler's conjecture.
- But **eliminating all posible irregular arregements** is very difficult, and this is what made the Kepler conjecture so hard to prove.



- 1831
- Carl Friedrich Gauss proved that the highest packing fraction that can be achieved by any packing of equal sphere.
- He proved that the Kepler conjecture is true if the spheres have to be arranged in a **regular lattice**.



- 2017
- Maryna Viazovska solved sphere packing problem in **8-dimensions** [1] (E_8 lattice). And in collaboration with others **24-dimensions** [2] (Leech lattice).
- **Winner of the Fields Medal 2022.**



Face-centered cubic structure

Theorem 1: No packing of congruent balls in Euclidean three space has density greater than that of the **face-centered cubic packing**, which corresponds to:

$$\rho = \frac{\pi}{3\sqrt{2}} \approx 0.7405$$

[1] M. S. Viazovska, "The sphere packing problem in dimension 8,"Annals of Mathematics, vol. 185, no. 3, pp. 991-1015, 2017.
[2] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. Viazovska, "The sphere packing problem in dimension 24," Annals of Mathematics, vol. 185, no. 3, pp. 1017-1033, 2017.

Face-centered Cubic Lattice

Definitions:

The generator matrix $\mathbf{M}$ has $v_1, \ldots, v_n$.

The Gram matrix $\mathbf{A} = \mathbf{M}^T\mathbf{M}$. Its entries (i, j) are given by $\langle v_i, v_j \rangle$.

The face-centered cubic (FCC) lattice has generator matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Example: The FCC has Gram matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Compute the Density of a Lattice

Density of a lattice in a fundamental cell:

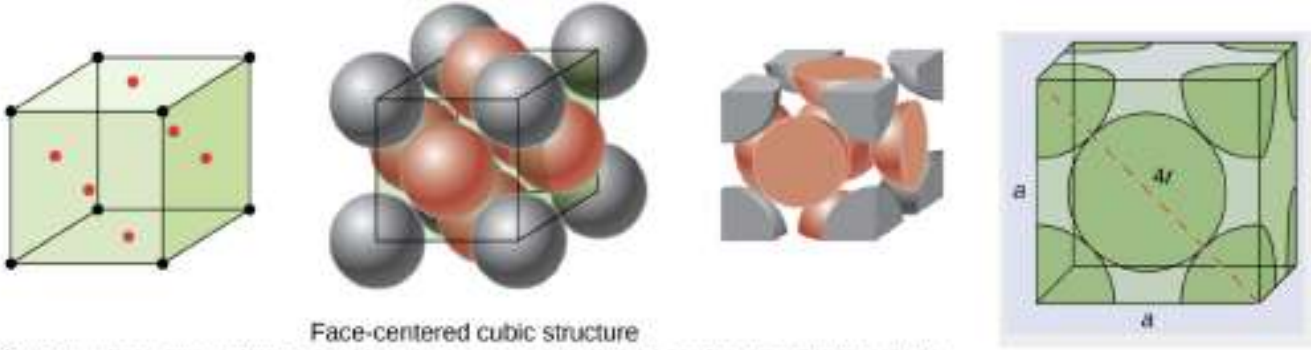
$$\frac{\mathrm{Vol}(B_r^n)}{\mathrm{Vol}(\mathbb{R}^n/\Lambda)} = \frac{\frac{\pi^{n/2}}{(n/2)!}r^n}{\sqrt{\det(\mathbf{A})}} \tag{4}$$

* Lattice volume: $\mathrm{Vol}(\mathbb{R}^n/\Lambda) = \sqrt{\det(\Lambda)} = \sqrt{\det(\mathbf{A})} = \det(\mathbf{M})$
* Sphere's volume: $\mathrm{Vol}(B_r^n) = \frac{\pi^{n/2}}{(n/2)!}r^n$, where $(n/2)!$ means $\Gamma(n/2+1)$
* Let $r = N(\Lambda)$ denote the minimal non-zero value of $\langle v, v\rangle$ among all $v \in \Lambda$.

n	1	2	3	4	5	6	7	8	24
Λ	A_1	A_2	A_3	D_4	D_5	E_6	E_7	E_8	Leech
due to		Lagrange	Gauss	Korkine-Zolotareff		Blichfeldt			Cohn-Kumar

Face-Centered Cubic (FCC) Density





Face-centered cubic structure

Figure 3. A *face-centered cubic solid has atoms at the corners and, as the name implies, at the centers of the faces of its unit cells.*

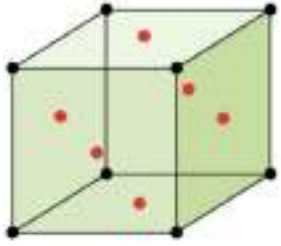
Let $r = N(A_3) = \frac{\sqrt{2}}{2}$ and $n = 3$

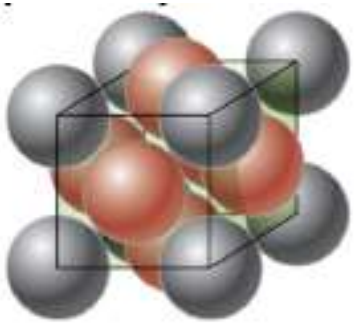
$$\frac{\text{Vol}(B_r^n)}{\text{Vol}(\mathbb{R}^n/A_3)} = \frac{\frac{\pi^{n/2}}{(n/2)!}r^n}{\sqrt{\det(\mathbf{A})}} = \frac{\frac{4\pi r^3}{3}}{2} = 0.74 \tag{5}$$

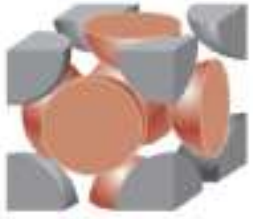
FCC include aluminium, copper, gold and silver.

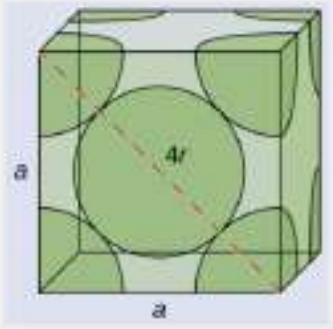
Face-Centered Cubic: Geometrical Calculations











Face-centered cubic structure

Figure 3. *A face-centered cubic solid has atoms at the corners and, as the name implies, at the centers of the faces of its unit cells.*

$$V_{\text{FCC spheres}} = \left(8 \cdot \tfrac{1}{8} + 6 \cdot \tfrac{1}{2}\right) \frac{4\pi}{3} r^3 \tag{1}$$

$$= \tfrac{16}{3} \pi r^3. \tag{2}$$

The diagonal of a face of the unit cell is $4r$, so each side is of length $2\sqrt{2}\,r$. The volume of the unit cell is therefore

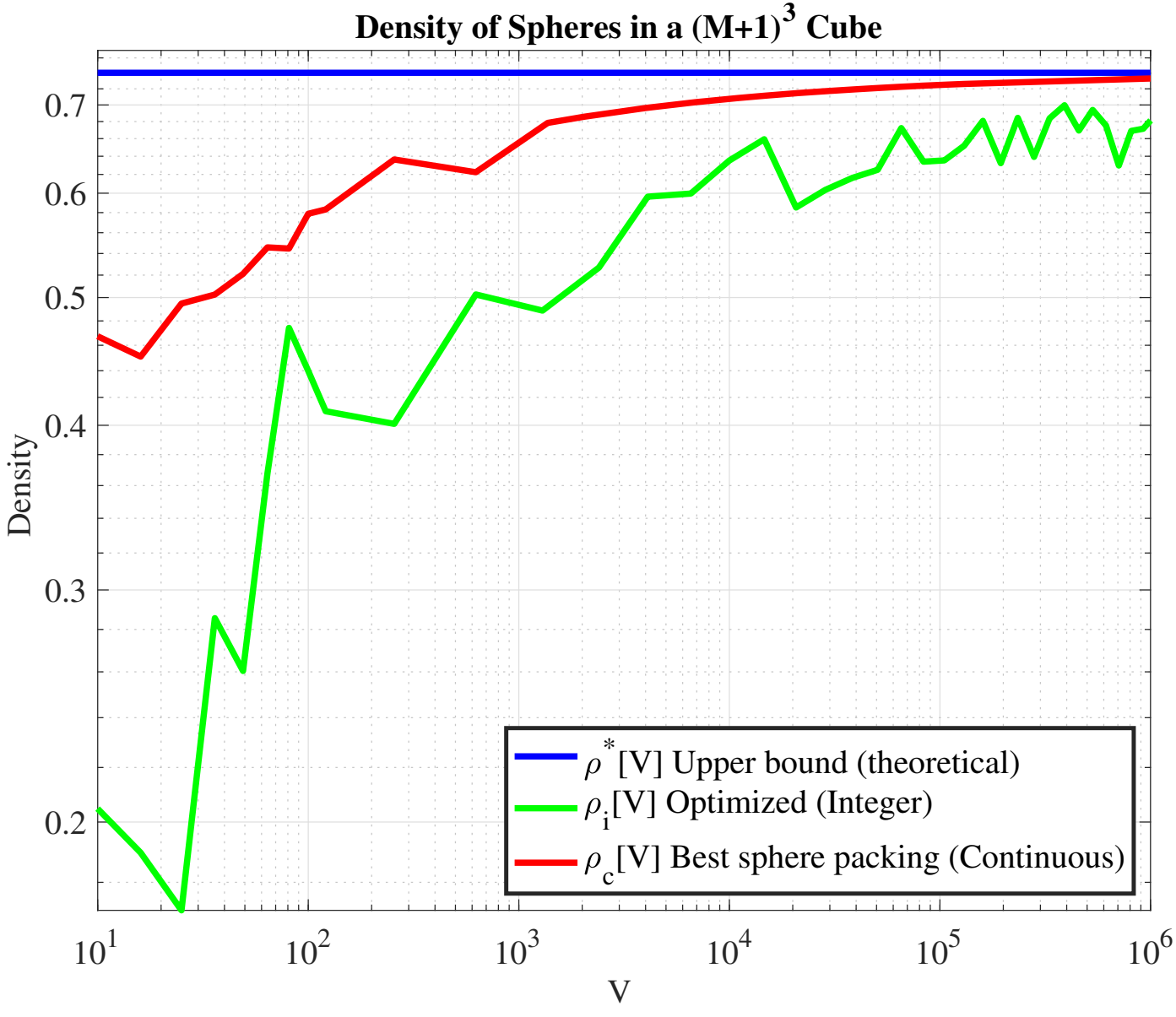
$$V_{\text{FCC unit cell}} = \left(2\sqrt{2}\,r\right)^3 = 16\sqrt{2}\,r^3, \tag{3}$$

giving a packing density $\eta = V_{\text{spheres}} / V_{\text{cell}}$ of

$$\eta_{\text{FCC}} = \frac{\pi}{3\sqrt{2}} \tag{4}$$

$$= 0.74048\ldots \tag{5}$$

Upper Bound for 3D-Sphere Packing



Spectral Imaging Applications

Application domains









Industrial machine vision
- Quality inspection
- Sorting
- Material detection

Medical
- Surgery
- Endoscopy
- Laparoscopy
- Smart pills
- Tissue regeneration monitoring

Agriculture
- Smart farming
- Weed control
- Disease detection

Automotive
- Night vision
- Autonomous cars
- Smart lights

Security
- Intrusion detection
- Camouflage detection
- Shipping monitoring
- Forensic investigation

New Space
- Global scale image analysis
- Environmental recovery
- Emission monitoring
- Event detection and disaster management

Environment
- Reforestation monitoring
- Oil spill detection
- Plastic detection

imec

public

Solving Sudoku and 3D Sphere Packing

Could you solve this puzzle?

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

[5] N. Diaz, A. Alvarado, P. Meza, F. Guzmán and E. Vera, "Multispectral Filter Array Design by Optimal Sphere Packing," in IEEE Transactions on Image Processing, vol. 32, pp. 3634-3649, 2023, doi: 10.1109/TIP.2023.3288414.

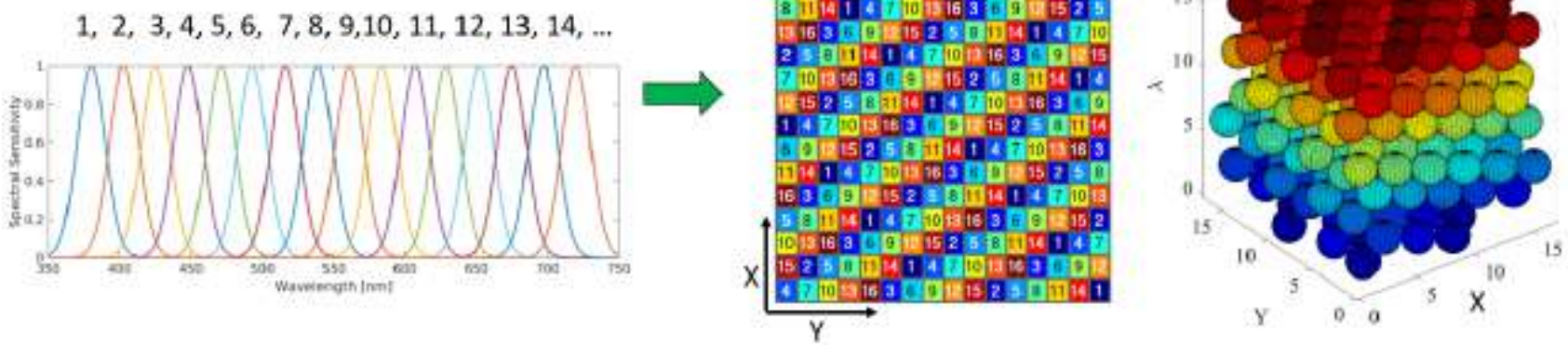
Solving Sudoku and 3D Sphere Packing

Could you solve this puzzle?

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

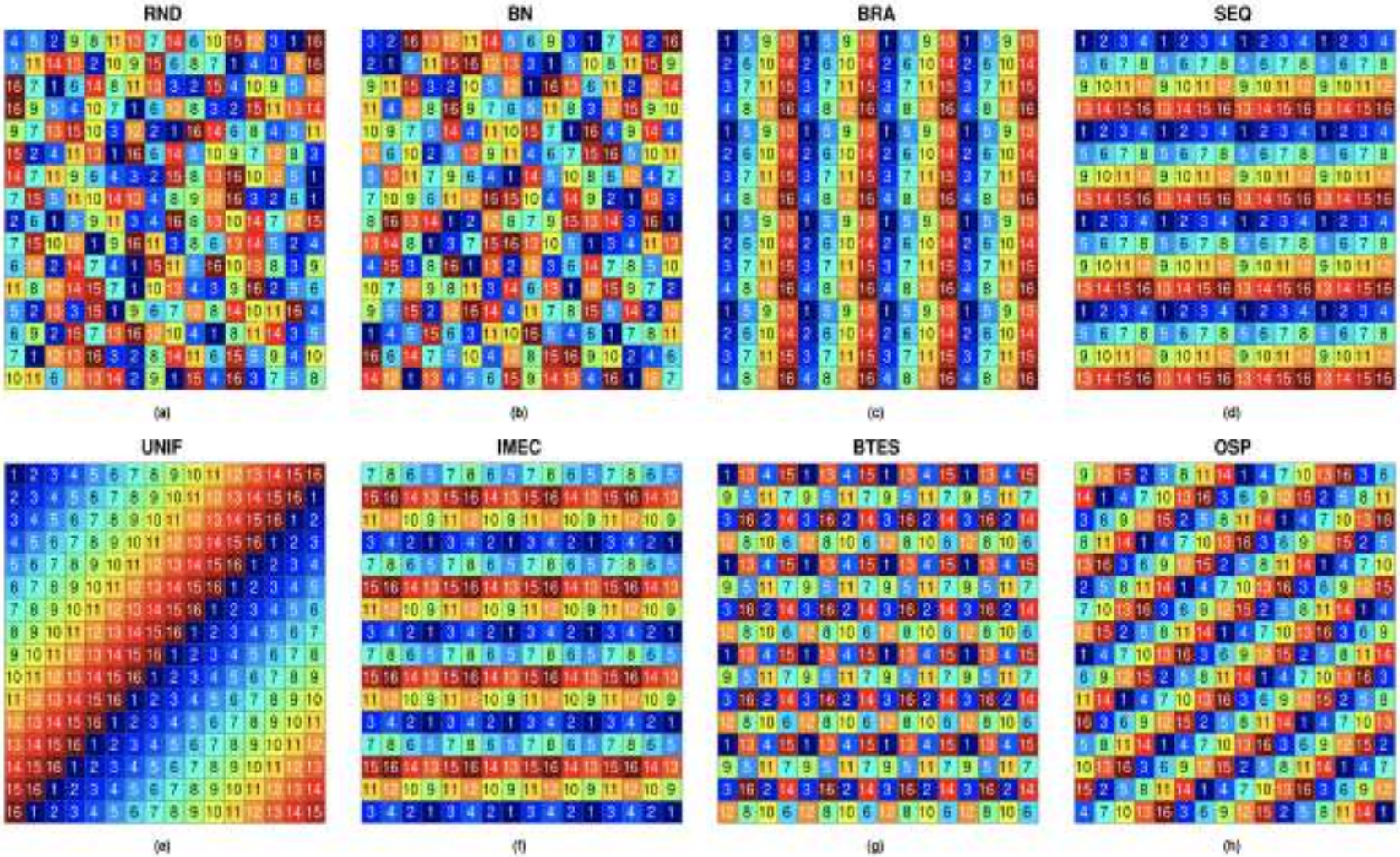
To assign numbers to the filters and distribute uniform along 3D space [8].



[5] N. Diaz, A. Alvarado, P. Meza, F. Guzmán and E. Vera, "Multispectral Filter Array Design by Optimal Sphere Packing," in IEEE Transactions on Image Processing, vol. 32, pp. 3634-3649, 2023, doi: 10.1109/TIP.2023.3288414.

Snapshot Spectral Imaging

* Multispectral filter array design [6].



[5] N. Diaz, A. Alvarado, P. Meza, F. Guzmán and E. Vera, "Multispectral Filter Array Design by Optimal Sphere Packing," in IEEE Transactions on Image Processing, vol. 32, pp. 3634-3649, 2023, doi: 10.1109/TIP.2023.3288414.

[6] A. Alvarado; N. Díaz; P. Meza; F. Guzman, E. Vera; , "Multispectral Mosaic Design using a Sphere Packing Filter Array", in imaging and Applied Optics Congress 2022.

Computation of Sphere Packing Density

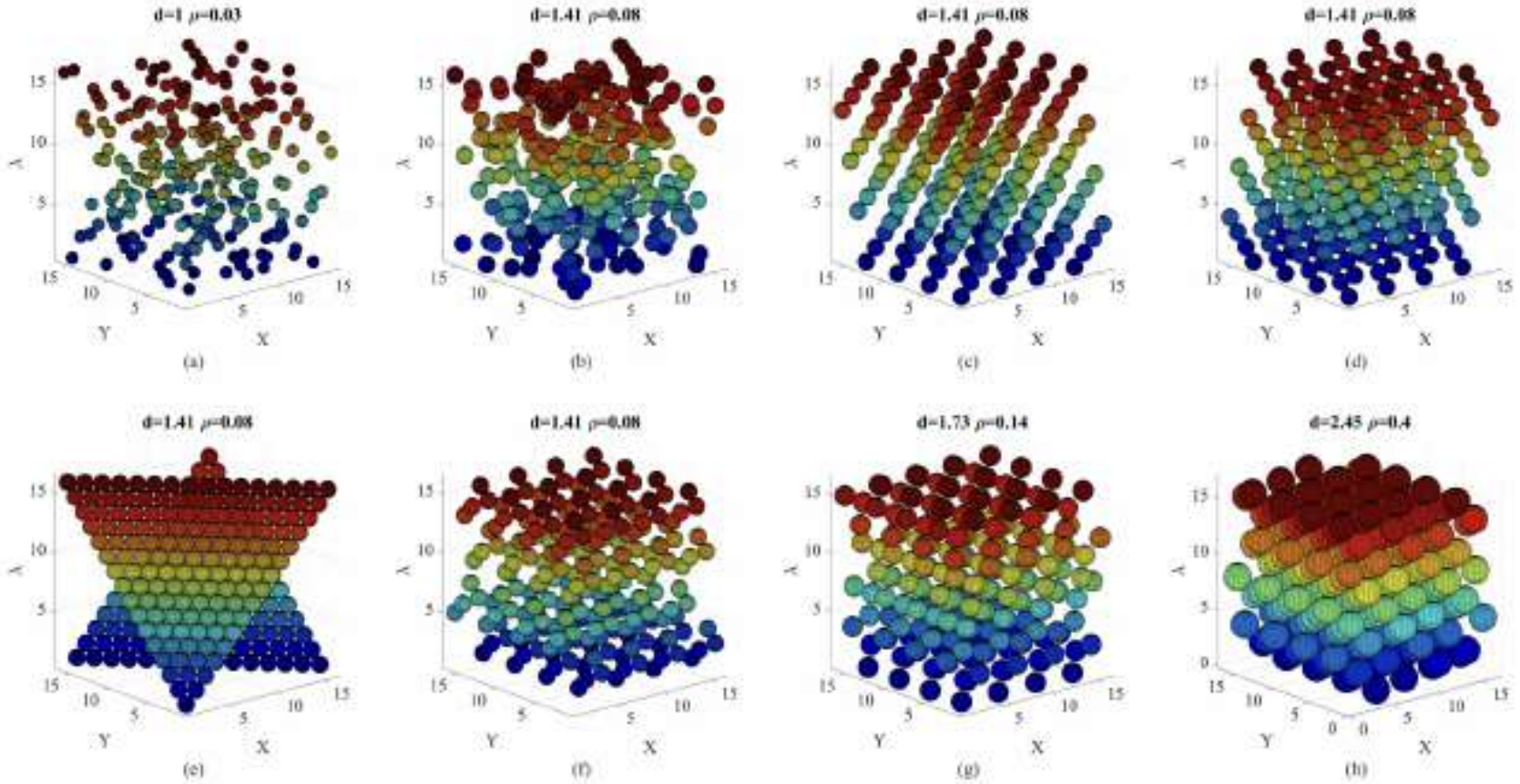
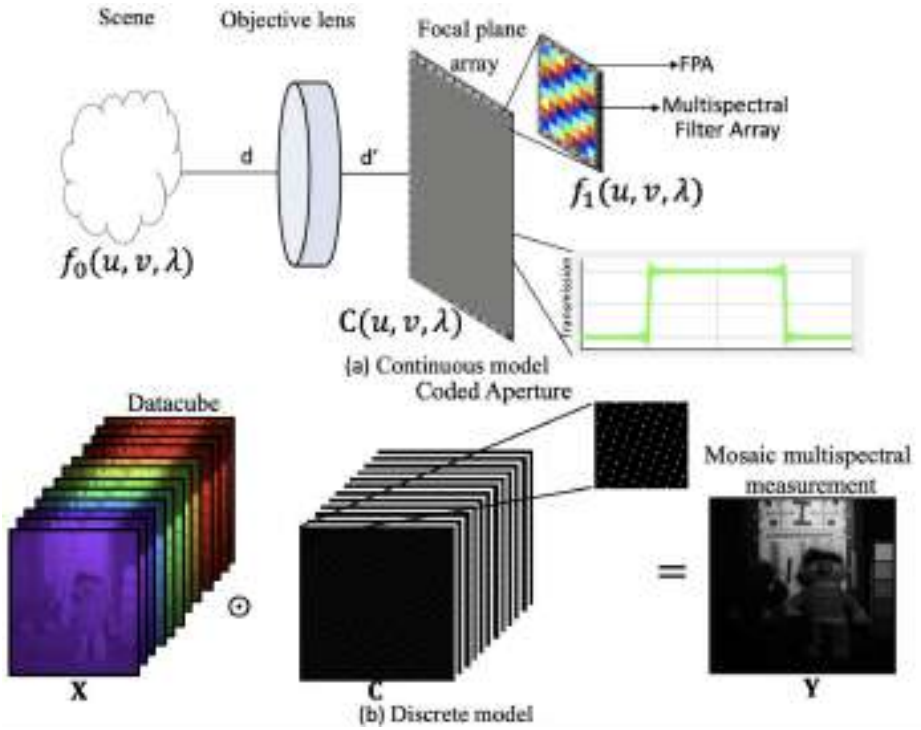


Fig. 4. SP density comparison of MSFAs coding patterns with 16 filters and no dominant filter. The first row depicts the MSFAs (a) RND, (b) BN CAs [12], (c) BRA [16], and (d) SEQ. The second row shows the MSFAs (e) UNIF, (f) IMEC [8], (g) BTES [5], and the proposed (h) MSFA-OSP computed using Eq. (12). State-of-the-art MSFAs experiments have corresponding sphere's diameter lower than the sphere's diameter of the OSP approach. The lower sphere's diameter is 1, with the SP density of 0.03 RND. Five state-of-the-art MSFAs have the same sphere's diameter and, therefore, the same SP density, corresponding to 1.41 and 0.08, respectively. The coding pattern BTES has sphere's diameter of 1.73 and a SP density of 0.14; the MSFA-OSP sphere's diameter is 2.45 and reaches a SP density of 0.4.

Snapshot Spectral Imaging

Snapshot spectral imaging[7, 8] $f(x, y, \lambda) \in \mathbb{R}^3$ are sampling problems in 3D.



Observation model.

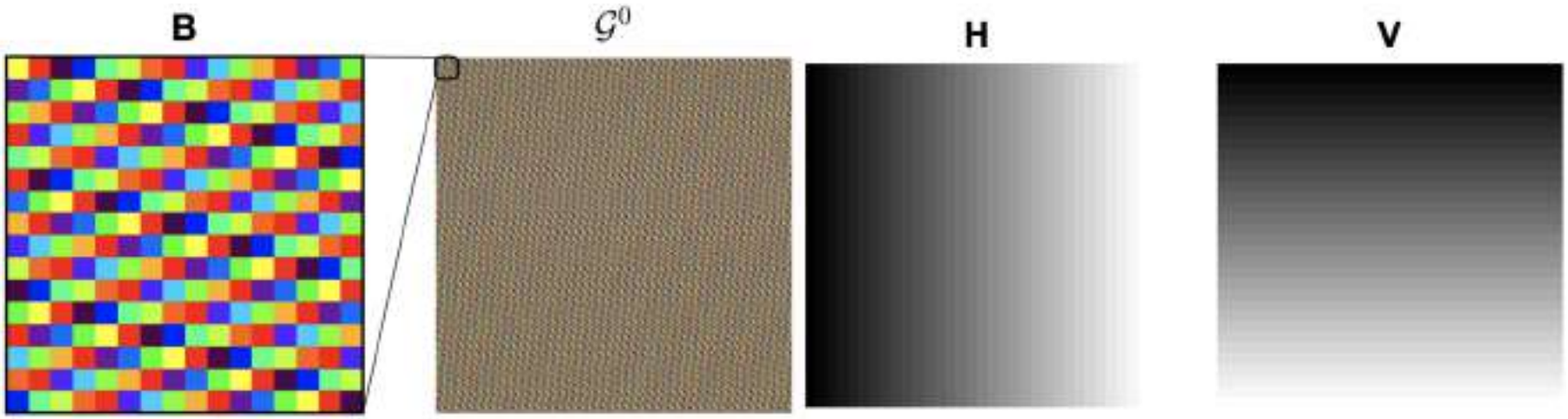
[9] N. Diaz, A. Alvarado, P. Meza, F. Guzmán and E. Vera, "Multispectral Filter Array Design by Optimal Sphere Packing," in IEEE Transactions on Image Processing, vol. 32, pp. 3634-3649, 2023, doi: 10.1109/TIP.2023.3288414.

[10] A. Alvarado; N. Díaz; P. Meza; F. Guzman, E. Vera; , "Multispectral Mosaic Design using a Sphere Packing Filter Array", in imaging and Applied Optics Congress 2022.

Computation of Multispectral Filter Array

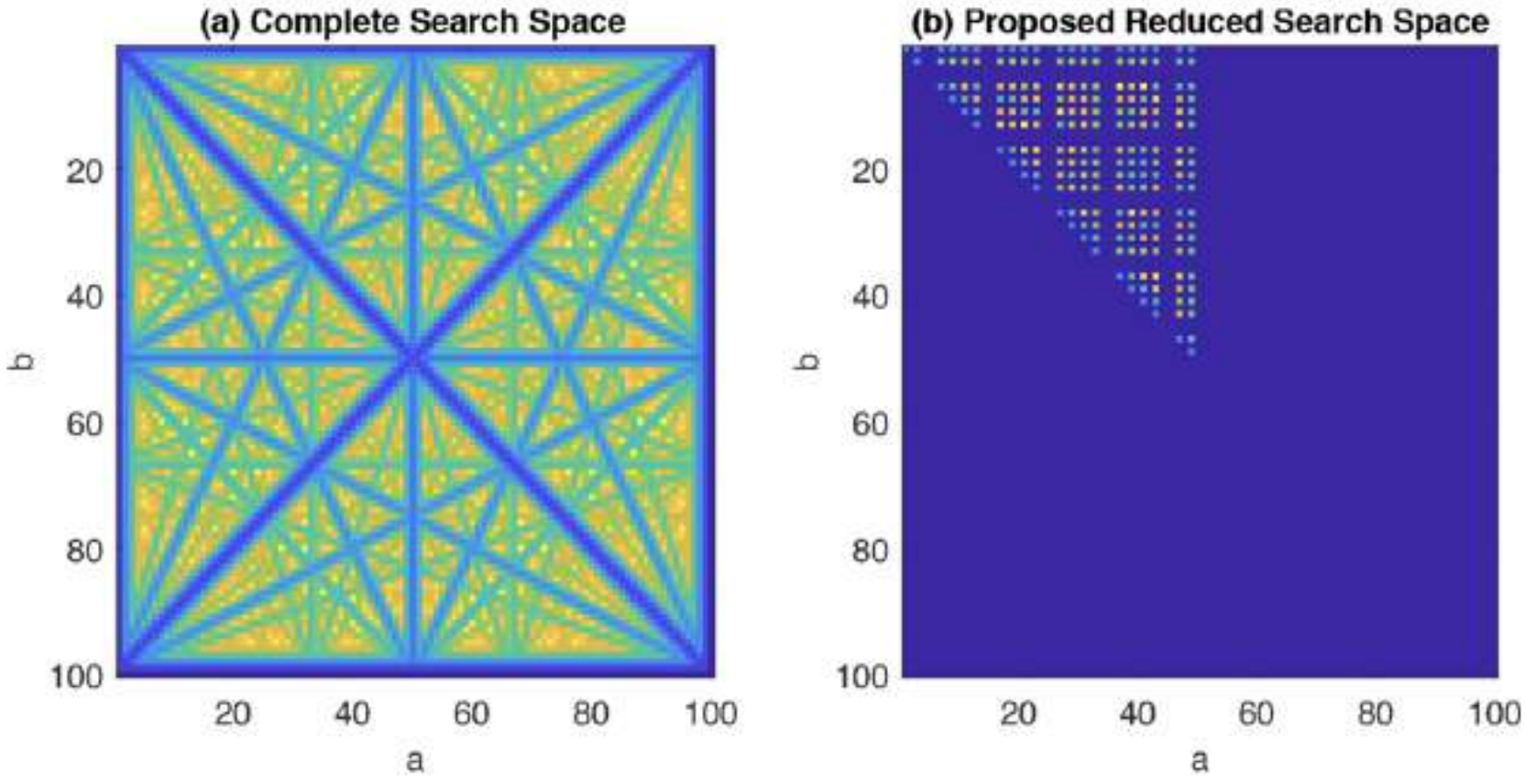
The sampling of video can leverage from the following solution to 3DN^2QP[11] to place the spheres within a 3D-container, $\mathbf{B}$ as follows:

$$\mathbf{B} = ((a \odot \mathbf{V} + b \odot \mathbf{H}) \mod K + 1), \tag{6}$$



[11] Allison, Lloyd and Yee, CN and McGaughey, M, "Three-Dimensional Queens Problems", Monash University, Department of Computer Science, 1989.

Search Space Algorithm of a and b



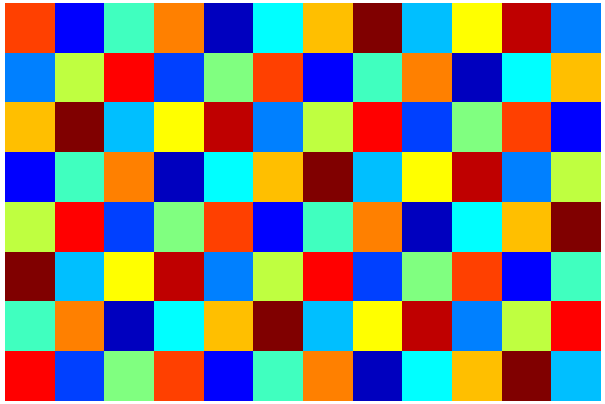
3D-Coded Aperture (CA) Sphere Packing Design

The positions of CUPOSP at the t^{th} frame are given by

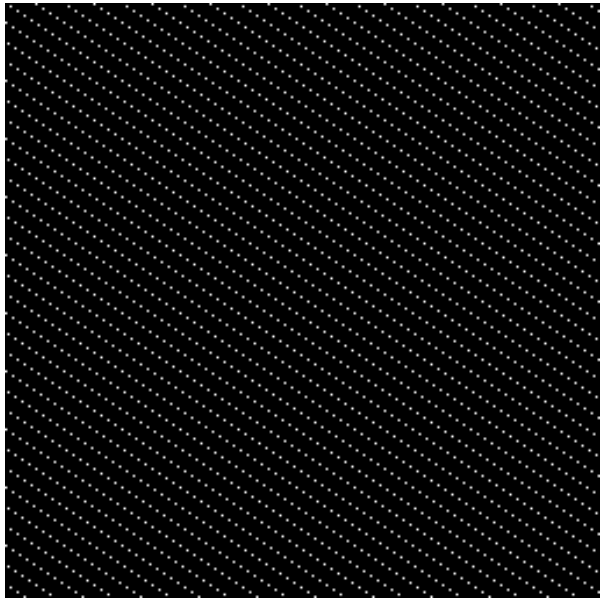
$$\mathbf{G} = \mathbf{A} \otimes \mathbf{B}, \tag{7}$$

where $\mathbf{A}$ is a matrix of all ones such that $\mathbf{A} \in \{1\}^{\alpha \times \beta}$, where $\alpha = \lfloor \frac{M}{K} \rfloor$, and $\beta = \lfloor \frac{N}{K} \rfloor$. The pattern $\mathbf{G}$ can be reorganized as CA

$$\mathcal{C}_{(i,j,t)} = \begin{cases} 1 & \text{if } t = \mathbf{G}_{(i,j)} \\ 0 & \text{if } t \neq \mathbf{G}_{(i,j)}, \end{cases} \tag{8}$$



CUPOSP $\mathcal{G}_{(:,:)}$.



coded aperture $\mathcal{C}_{(:,:,t)}$.

Compute Spheres Distance

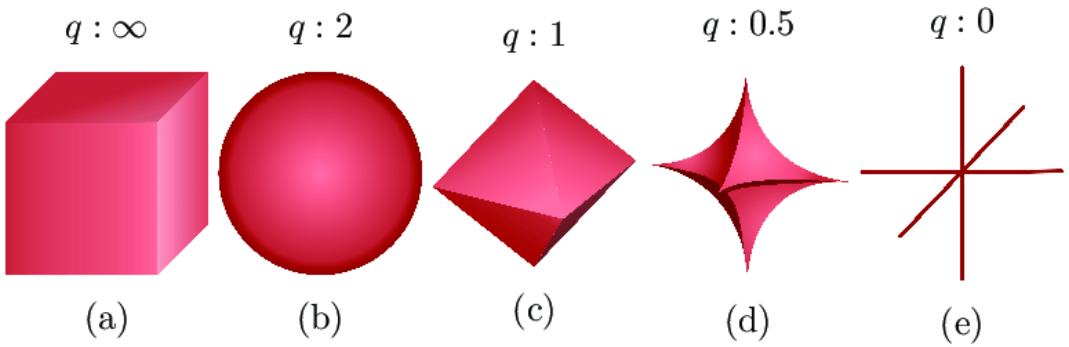
The resulting matrix $\mathbf{G} \in \mathbb{R}^{M \times N}$ can be reorganized as $\mathbf{p}_l = [i, j, \mathbf{G}_{(i,j)}]$, where $\mathbf{P} = [\mathbf{p}_1, \dots \mathbf{p}_l \dots \mathbf{p}_V] \in \mathbf{R}^{3 \times V}$, with indexes $i, j \in \{1, \dots, K\}$, where $V = K^2$ is the number of spheres. Thus, the distance function of V spheres is

$$d^*(V) = \max(\min_{1 \le l_1 < l_2 \le V}, D_{l_1, l_2}), \tag{9}$$

where $D_{l_1,l_2} = \|\mathbf{p}_{l_1} - \mathbf{p}_{l_2}\|_2^2$.

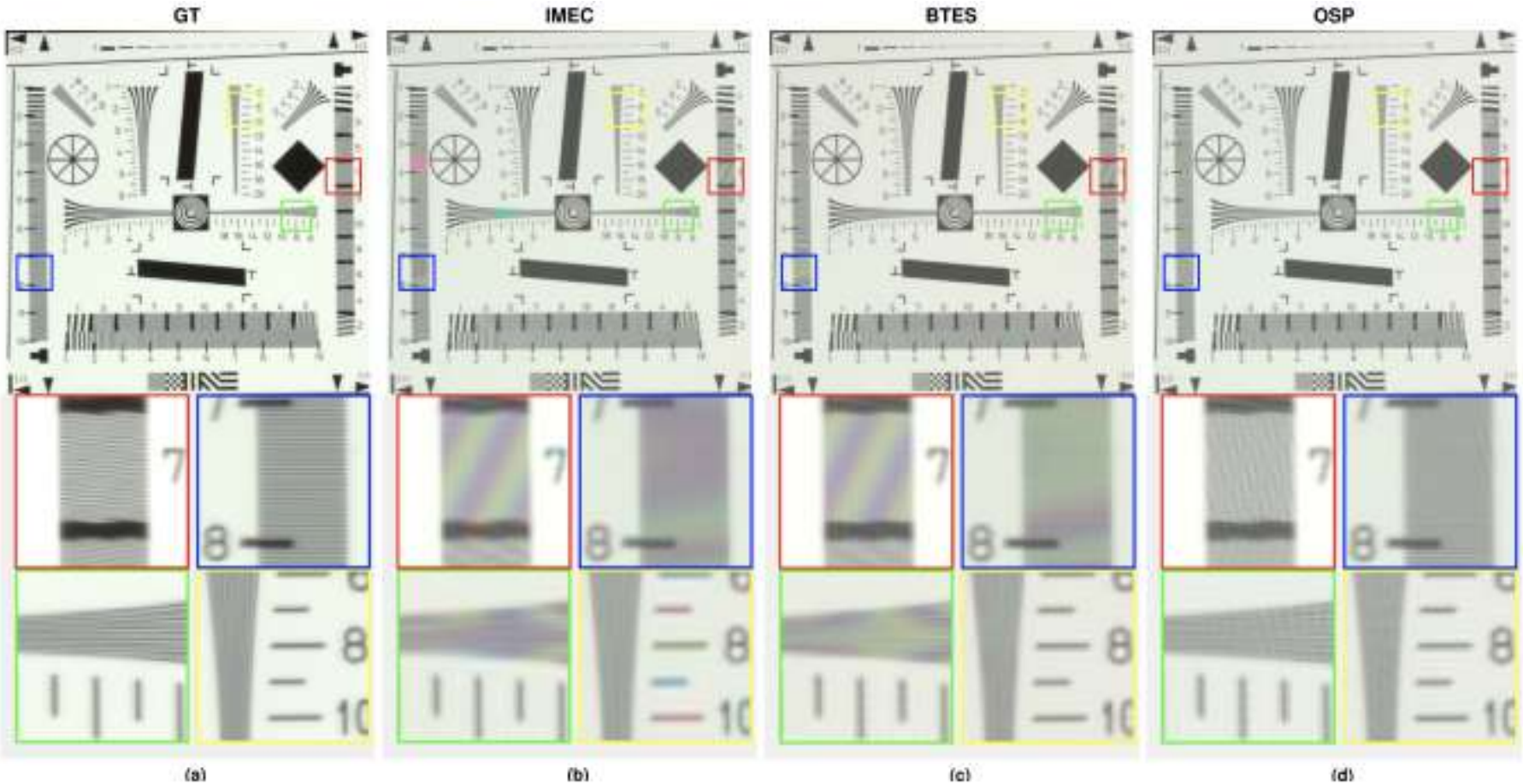
Why Spheres Instead of 3D Voxels?

To measure the distance between pairs of sampling points, we use the ℓ_2 norm, where $q = 2$, resulting in a sphere packing optimization when the pairwise distance is computed $D_{l_1,l_2} = \|\mathbf{p}_{l_1} - \mathbf{p}_{l_2}\|_2^2$



Snapshot Spectral Imaging

* Spectral Reconstructions $f(x, y, \lambda) \in \mathbb{R}^3$, the sphere packing reduce the zipper effect and color distortion.



Snapshot Spectral Imaging

* Ximea camera with the IMEC multispectral filter array.



Spectral range 460-600 nm

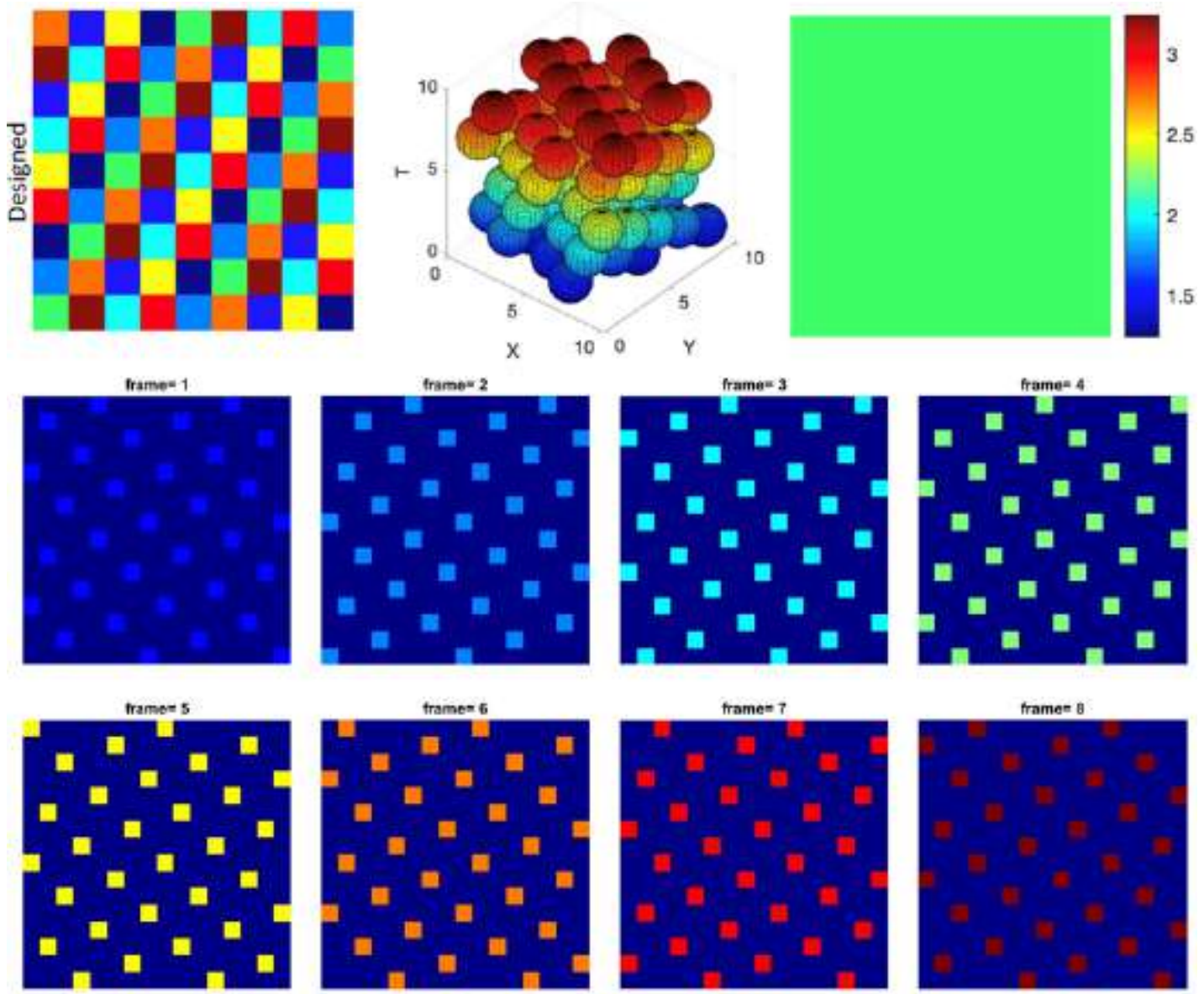
$2/3$" CMOS, Hyperspectral Snapshot sensor

Scans: up to 170 $cubes/sec$

16 Bands

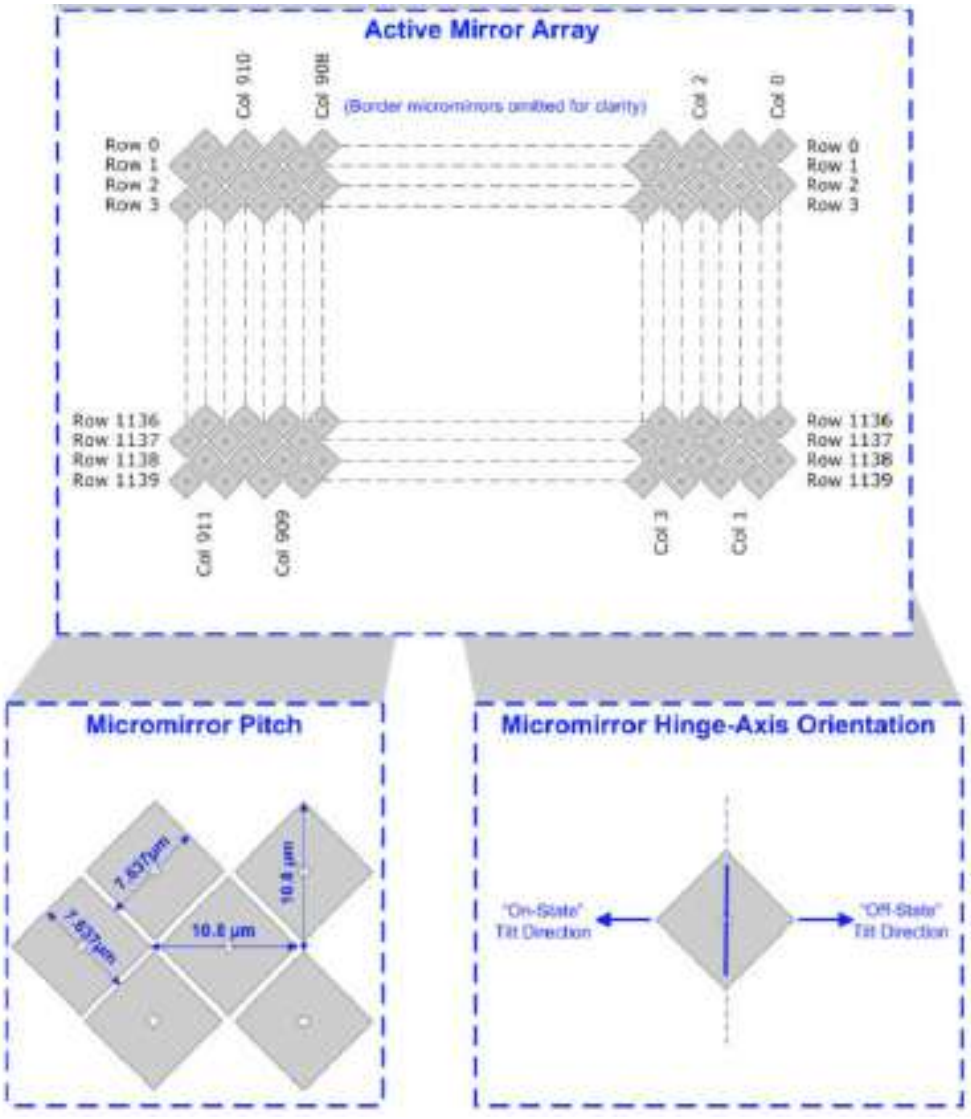
Coded Aperture Design

We solve the temporal dimension



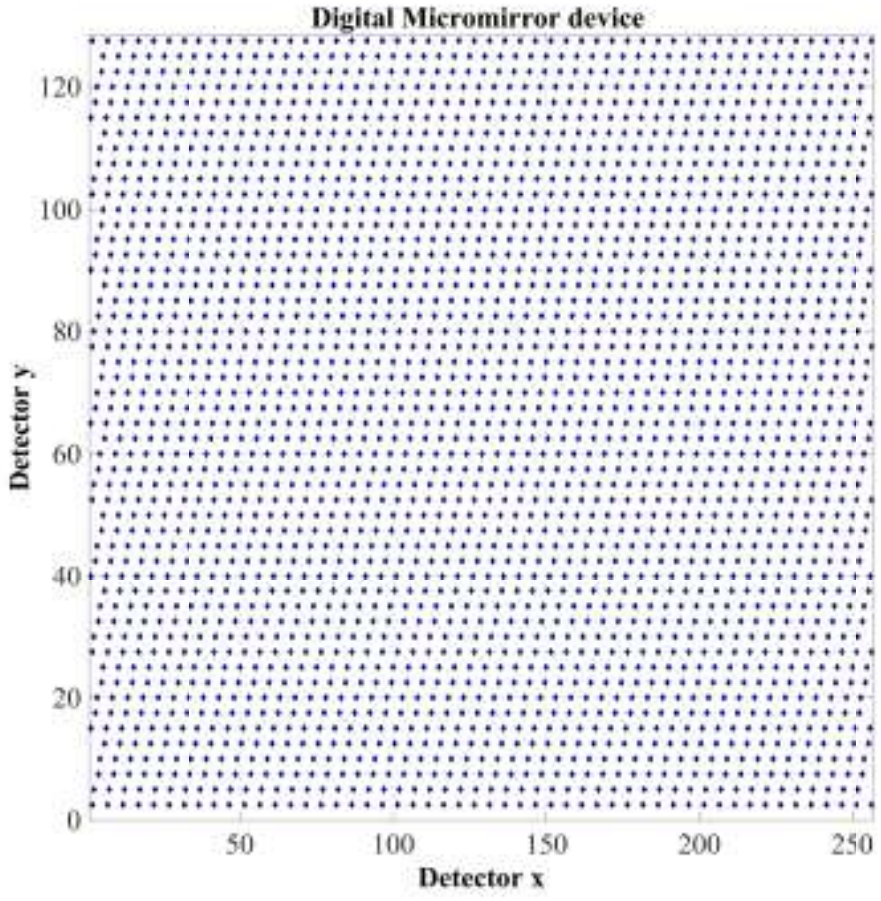
Diamond DMD Structure

Micromirror Array, Pitch, and Hinge-Axis Orientation



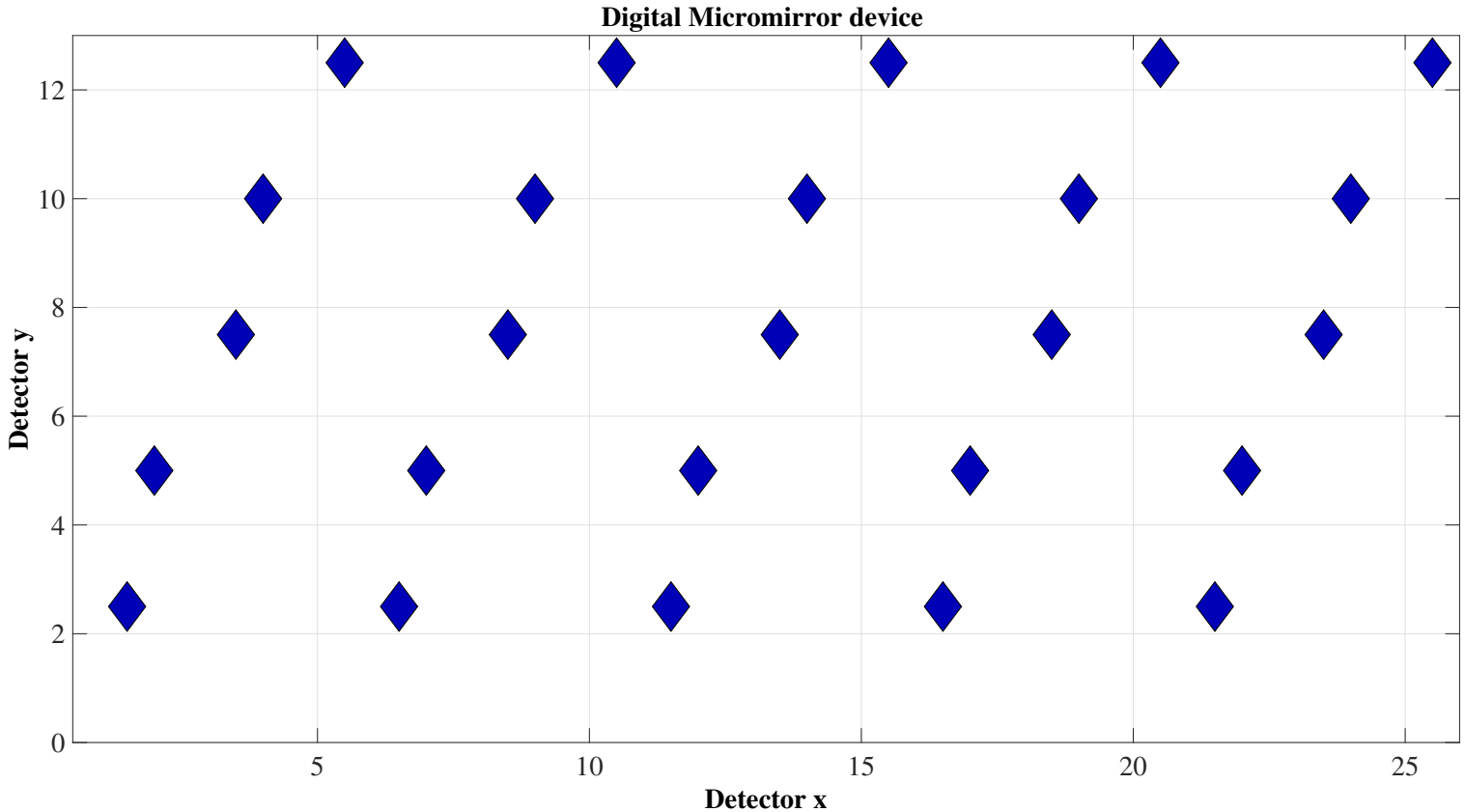
Example Sampling Position 256×256

One single frame of diamond DMD 256×256, Transmittance $= 1/25$



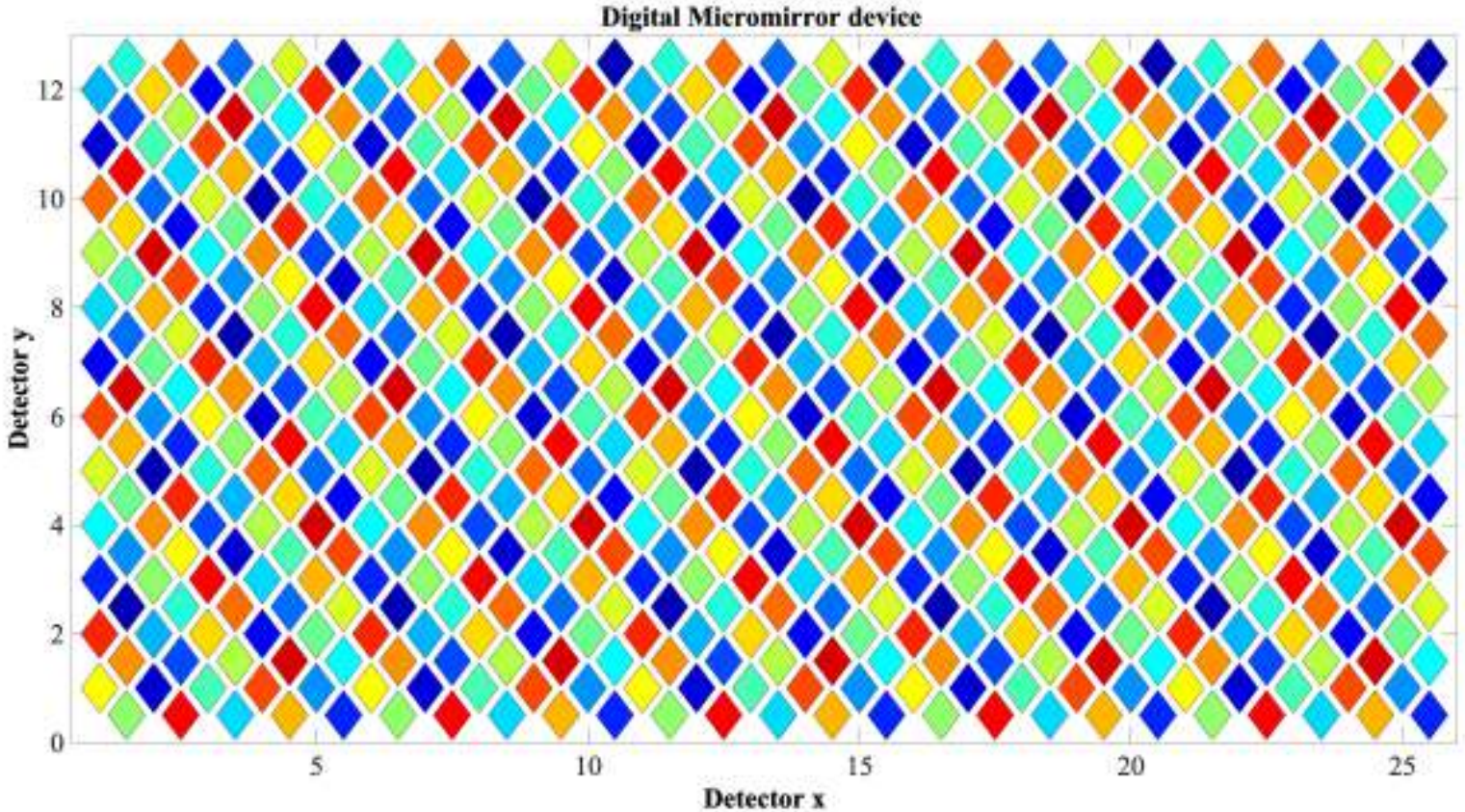
Zoomed Version: 1 kernel

One single frame of diamond DMD 25×25, Transmittance $= 1/25$



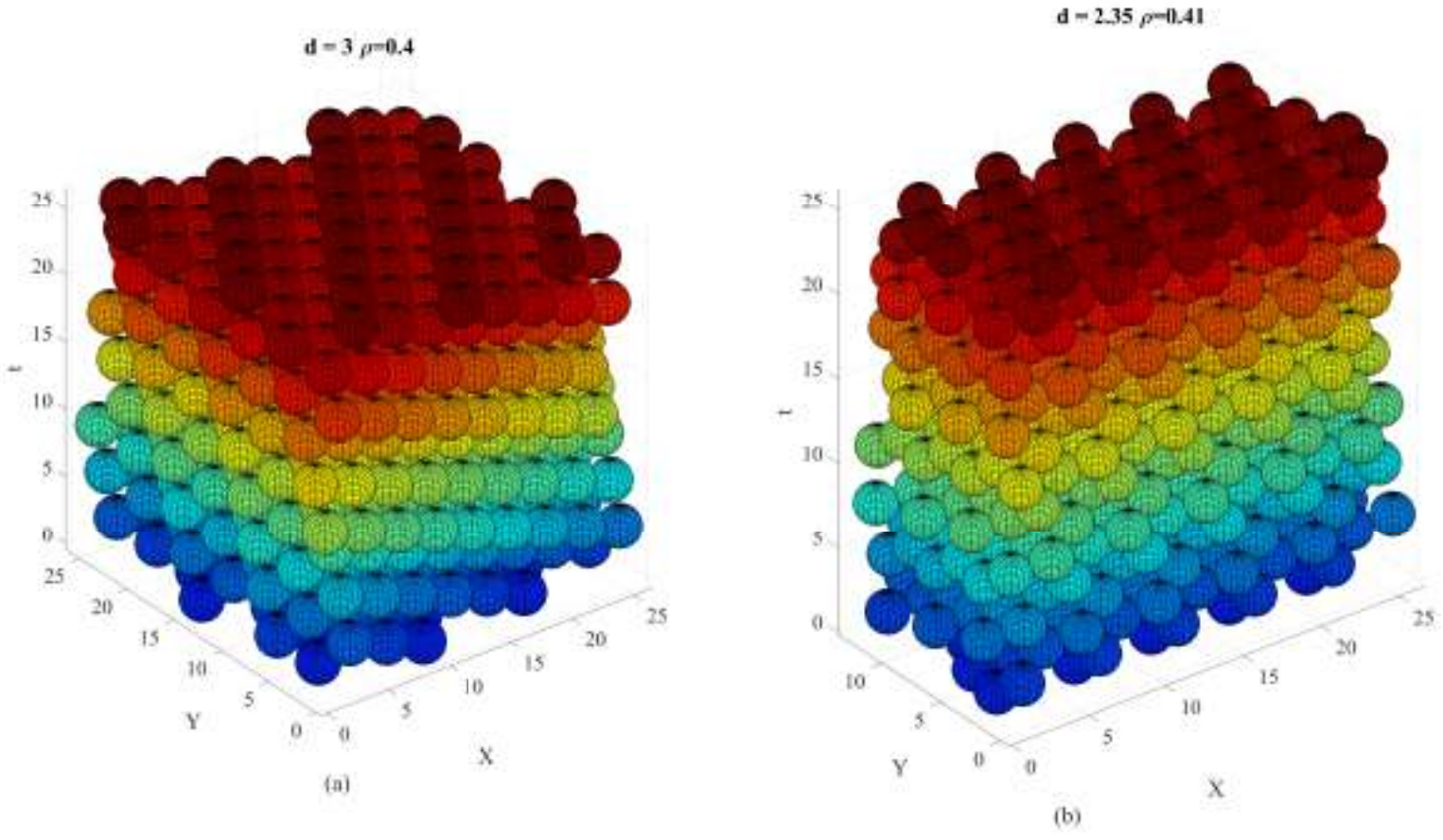
Example of All Sampling Positions: 1 kernel

25 different sampling positions Diamond DMD 25×25



Grid DMD Against Diamond DMD

Comparison of SP densities between (a) Grid DMD and (b) Diamond DMD



Sphere Packing Density Comparison

Comparison of SP densities between Grid DMD and Diamond DMD.
Where V is the number of spheres.

More simulations with higher number of sphere are required.

